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Assimilating Mathematical Thinking to the Learning of Shadows

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Abstract

This study focuses on a teaching experiment with 33 six-graders in a Kearny public school in Hudson County, New Jersey, during the 2017-2018 academic year. More specifically, this study explored a) the types of tasks and tools that can be used to develop students' covariational and correspondence reasoning in learning about shadows and b) the nature of students' reasoning about covariation and correspondence relationships as students engage in the use of tools and tasks. The results showed that the simulation and the tasks I designed had the students engaged in the learning process. Students were able to reason about the characteristics of an object's shadow, construct covariational relationships between the angle of the sun and the length of the shadow for objects and reason about the correspondence relationships between the height of objects and the length of their shadows. The study presented here was intended to explore the ways that we could incorporate simulations into students' learning of shadows and time that would help develop students' mathematical thinking of ratio and proportion.

Keywords: correspondence and covariational relationships, NetLogo, ratio and proportional reasoning, shadows and time, simulations

MONTCLAIR STATE UNIVERSITY

Assimilating Mathematical Thinking to the Learning of Shadows

by

Taheeda Shwana Street-Conaway

A Master's Thesis Submitted to the Faculty of

Montclair State University

In Partial Fulfillment of the Requirements

For the degree of

Master of Science

January 2019

College of Science and Mathematics

Mathematics

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ASSIMILATING MATHEMATICAL THINKING TO THE LEARNING OF SHADOWS

A THESIS

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Montclair, NJ

2019

ACKNOWLEDGEMENTS

My educational experience has been greatly enhanced by the structure and support of the staff of the Mathematics and Science department at Montclair State University. Working with the staff in a professional learning community (ACMES Project) has provided an essential network. Deepest thanks to Dr. Nicole Panorkou, Dr. Mika Munakata, and Dr. Teo Paoletti for their special support and help. This research is supported through a STEM+Computing grant from the Division of Research on Learning of the National Science Foundation (# 1742125). The reviews expressed do not necessarily reflect official positions of the Foundation.

I would like to thank: my fellow MSU students and partners for providing a sounding board for ideas, moral support and a sense of community. I would like to thank Sowmith Etikyala for partnering with me in creating the NetLogo simulation. Without his computer programming skills, the simulation would not have been successful. Both Erell Feb Germia and Debasmita Basu for critically guiding my thinking. Erell for inspiring me to think critically when it came to student mathematical thinking and providing her support; Debasmita for her support with facilitating the video recordings and interviewing of the students.

Finally, I would like to express my deepest appreciation to my loving husband, Ake Street-Conaway, and daughter, Jasmine Street-Conaway, whom both have been incredibly supportive of my educational pursuits. Both have spent countless hours listening to my mathematical and educational theories.

Author

Taheeda Shwana Street-Conaway

TABLE OF CONTENTS

LIST OF FIGURES

Assimilating Mathematical Thinking to the Learning of Shadows

1. The Literature on the Learning of Shadows

1.1 Incorporating mathematical thinking into science learning

A type of active learning based on students' direct engagement with research is *inquiry-based learning*, which refers to the “forms of learning driven by a process of inquiry” (Healey, 2005, p. 7). In their study, Edelson, Gordin, and Pea (1999) explored the opportunities and obstacles presented by scientific visualization technology to support inquiry-based learning. Edelson et al. (1999) believed that a learner who participates in inquiry experiences would gain and develop general inquiry abilities, acquire specific investigation skills, and understand science concepts and principles. They discussed elements which address different facets of the challenges of implementing inquiry-based learning in a classroom, which included creating and making investigation techniques accessible and designing to engage learners in an inquiry task in which the learner can achieve mastery of the skill. In other words, students develop new content understanding through discovery and refinement.

STEM (Science, Technology, Engineering, and Mathematics) education aims to engage students in inquiry-based learning through the integration of science, technology, mathematics, and engineering. For several years, politicians and educational leaders have been working to strengthen STEM education in the United States. Thomasian (2011) explains how the United States has fallen behind in truly understanding the benefits of STEM education. Thomasian noted that although STEM career opportunities are expected to grow 17% between 2008 and 2018, many higher education institutions had not increased their output. Secondary school administrators are not likely to expand a disciplinary offering in an already crowded school curriculum, even when educational leaders believe it to be important (Catterall, 2012). Daugherty, Carter, and Swagerty (2014) stated that most K-12 teachers had not been trained to integrate relevant STEM topics into their classroom teaching and curriculum materials. Consequently, there is a need to develop integrated STEM materials that are aligned with the existing curriculum so that both administrators and teachers see the connections between these materials and what is currently being taught in schools.

This study aimed to construct tasks and tools that would engage students in inquiry-based learning and would integrate math, science, and technology as content for that learning. The content chosen for this study is the learning of shadows for science and ratio and proportional reasoning for mathematics. The specific concepts were chosen because they are concepts that are currently being taught in the science and math curriculum and because students face difficulties in understanding these concepts. The following paragraphs describe how shadows are currently taught in schools and give an overview of the research on ratio and proportion and how this can be used for teaching shadows.

1.2 The learning of shadows

Everyone notices the differences that occur in shadows over the span of a day and during the change of seasons. Most children are fascinated by their own shadows: how their shadows grow longer towards the end of a day, how their shadows are so short at noon, how their shadows move with them and mimic whatever they do.

In discussing the importance of what he calls "backward design", Grant Wiggins (2005) identified three fundamental features of the educational process: learning goals, assessment, and instruction. Using the backward design the teacher would first set the *learning goals* of shadows by identifying valued outcomes appropriate for the students and their grade level. Second, the teacher would create an *assessment* aligned to the learning goals that were set. The assessment could be in the form of an activity to provide information on the extent of student achievement of the learning goals. Such an activity would allow the students to explain what they think about the position of the sun if a shadow was short, or what happened to the position of the sun if the shadow was under the object. Third, the *instruction*, teachers plan the most appropriate lessons and learning activities to address the learning goals. In the form of modeling basic skills for learning. However too often, the teaching may focus primarily on presenting information without extending the lessons to help students make meaning or transfer the learning.

To provide any evidence that the student had obtained the information when teaching phenomena in science, the investigations during *instruction* (Wiggins' third fundamental feature) would require careful observation, record keeping, and the development of explanations and mental models by the students (Rice, Zachos, Burgin & Doane, 2005). Typical instruction of the teaching of shadows involves observing the sun and an object's shadow over time and recording the data in order to develop an understanding of the relationship formed between the position of the sun and an object's shadow. This process could sometimes take days because in order for the students to have an accurate account of the relationship between the sun and an object's shadow the data collection needs to be at different times of the day and accurate.

This study's goal is to engage students in dynamic experiences of learning about the relationships between shadows and time by exploring a simulation as a way to conceptualize the relationships between the position of the sun and the direction and length that the shadow cast. Visualizations allow students to investigate complex phenomena that they cannot directly observe (Pallant, Lee & Pryputniewicz, 2012). Therefore, I believe simulations will allow students to make predictions, test hypotheses and observe outcomes of natural phenomena.

1.3 The Mathematical Relationships of Shadows

Science teachers face a multitude of challenges when assimilating mathematical thinking into their classrooms, including mathematical misconceptions and incorrect ideas resulting from students' misunderstanding about a mathematical idea or concept (Dougherty, Bryant, Bryant & Shin, 2016). The learning of shadows involves engaging students in constructing relationships about shadows, such as expressing a relationship between the angle of the sun at different times of day and the length of shadow as well as the relationship between the length of the shadow and the length of the object that casts a shadow. The latter is a linear relationship and relates to students' understanding of ratio and proportion.

Proportional reasoning is a milestone in students' mathematical development (Cramer & Post, 1993). Proportional reasoning is when two quantities are related by a proportionally, which involves the use of multiplicative relationships to compare quantities and to predict the value of one quantity based on the values of another and transfer to new situations (Lobato, Ellis & Zbiek, 2010). While the importance of ratio and proportional reasoning is regularly emphasized, in middle school mathematics, the teaching and learning of these concepts continue to present challenges (O'Keeffe & White, 2018).

One challenge is making sure the student understands the difference between additive thinking and multiplicative thinking when dealing with ratios (Dougherty, et al. 2016). Siemon, Bleckly, and Neal (2012) stated that one reason students experience difficulties with ratio and proportion is that many students intuitively apply additive strategies rather than using multiplicative thinking. Multiplicative thinking is the foundation of proportional reasoning, and proportionality may be the unifying theme needed to highlight the important mathematics of one's middle school years (Lanius & Williams, 2003), therefore it is important to find ways to help students reason multiplicatively.

A lack of deep understanding involves incorrectly applying a procedure or an algorithm students learned by rote memorization. For example, when solving a proportion to find missing value, students are taught to apply the method of cross multiplication. Proportional reasoning is not usually involved when students use memorized rules or algorithms (Lamon, 1999). Consequently, it is important to develop students' conceptual understanding of proportion.

1.4 Covariation and correspondence relationships

The most important type of thinking required for proportional reasoning is the capability to analyze change. An alternative way of thinking about ratio and proportional reasoning and analyzing change are the correspondence and covariational types of reasoning. A correspondence (Smith, 2003) reasoning involves viewing a function as a fixed relationship between two sets. This way of thinking allows students to solve problems when given a graph, construct and interpret graphs, tables, and find missing values. As Confrey and Smith (1995) argue "conventional secondary functions curriculum is dominated by a correspondence approach where a function is described via a rule (e.g., f(x) = 3x+2)" (p. 67).

Smith (2003) and Smith and Confrey (1994) offered an alternative to the correspondence point of view, called the covariation approach. Covariational reasoning defined by Carlson et al. (2002) is, "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). With this approach, students examine a function in terms of a coordinated change of *x*- and *y*-values. Panorkou and Maloney (2016) describe this as a relationship between quantities of two sequences of data, where one pattern changes at the same time as a quantity in the other pattern.

Two types of covariational reasoning were used in this study: non-numeric and numeric. The first type, non-numeric, allows the students to analyze the relationship between the sun's position and the position of the object's shadow, as well as to identify the factors that change over time (the sun's position, the object's shadow length, and direction) without the use of numbers. The second type, numeric, allows the students to discuss numerically the length of an object's shadow in relation to the suns position in the sky. For example, the sun changes its position 15 degrees each hour which causes the object's shadow to change in length every hour. These relationships can be shown using arrows on the tables (Figure 1).

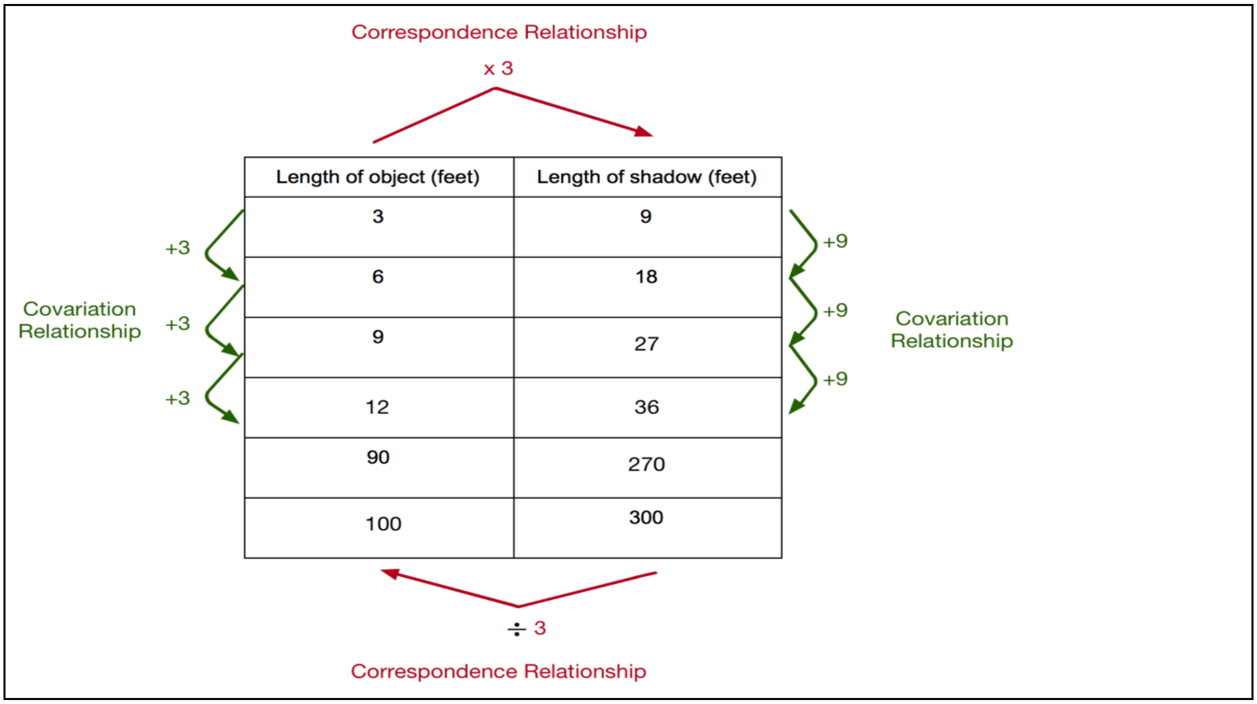


Figure 1: Correspondence and covariation relationship table.

I believe that engaging students in covariational and correspondence reasoning would not only help them identify the proportional relationships in the learning of shadow but would also help them distinguish between linear and non-linear relationships.

1.5 NetLogo as a means to explore mathematical relationships

Willis (2010) notes that eliminating mathematical misconceptions is difficult, and merely repeating a lesson or providing extra time for practice will not help. Willis (2010) also states students need tangible experiences to avoid these misconceptions. The National Council of Teachers of Mathematics (1991) stated that meaningful tasks can increase student motivation and also challenge students to think about the relevant mathematical concepts. I take this to imply those meaningful tasks should allow for collaboration and fun and at the same time help students construct relationships about the quantities involved in the situation. Utilizing a simulation to model a real-life phenomenon can illustrate the quantitative relationships dynamically and provides the elements to explore, navigate and obtain more information about the event that could not be acquired from mere experimentation.

Designed by Uri Wilensky in 1999, NetLogo is a programmable modeling agent software intended for simulations of nature; it is freeware and can be downloaded from http://ccl.northwestern.edu/netlogo/. NetLogo supports work in many fields, which include dynamic demonstrations of the real world, to determine ways to represent real-world situations mathematically and enables students to recognize common structures that may often be ignored in the traditional mathematics classrooms (Wilensky & Reisman, 2006). NetLogo includes a growing Model Library of examples of science and mathematical models which allows users to explore many of the constraints of authentic phenomena (ie. sunrise and sunset).

NetLogo has the potential to engage students in mathematical modeling, which uses mathematics as a tool to study phenomena (Carreira, 2001). Due to its potential, NetLogo is the ideal platform for students to use in order to study the characteristics of shadows. This platform provides the students with an accurate account of the relationship between the sun and an object's shadow over the course of a day. The mathematical modeling experiences provided by NetLogo models engage students in doing mathematics while maintaining fidelity in mathematical content, cognition, and pedagogy (Johnson, 2007). Research shows that visualizations allow students to investigate complex phenomena. NetLogo provides simulations to model complex phenomena while at the same time NetLogo encourages students to think mathematically. Through the use of NetLogo simulations assisting in the learning, process students will become more efficient in ascertaining the complex phenomena shadows and time. Consider the smallest amount of excitement, students can gain an opportunity to explore the relationships of shadows in ways that facilitate a higher level of cognitive demand in a realistic context.

2. Method

A design based research study was used in the study to answer the following questions:

a) What types of tasks and tools can be used to develop students' covariational and correspondence relationships as they learn about shadows?

b) What is the nature of students' reasoning about covariational and correspondence relationships as they engage with those tasks and tools?

A design based research is a research methodology in which a series of tasks are purposefully created to make changes to and, at the same time, observe student responses. The purpose of design studies is to develop a class of theories about the process of learning and the means that are designed to support that learning (Cobb, Confrey, DiSessa, Lehrer & Schauble, 2003).

In a whole class teaching experiment, the classroom teacher collaborates with the research team to give guidance to the students. Although a whole classroom experiment was used in this study, our video recordings were focused on the same several pairs of students (of Kearny public school in Hudson County, New Jersey) over the course of a week. The aim was to create a small-scale version of a learning ecology so that it could be studied in depth and detail (Cobb & Steffe, 1983). For the design based research study, I designed a module for teaching and learning shadows and time. The module consisted of three investigations and a NetLogo simulation that accompanied two of these investigations.

2.1 The simulation

I designed a simulation on NetLogo to show the behavior of an object's shadow over time. This simulation illustrates how shadows are created by the light of the sun on different objects over a 12 hour period (Figure 2). The user may observe how shadows change by manipulating time and switching objects. The simulation aims to help the user understand how the variables (angle of the sun, length of an object, length of shadow) work together to create shadows; for example, by recognizing that the length of the resulting shadow depends on the angle of the sun in the sky. The time slider allows the user to change the angle of the sun by 15 degrees per hour from 6 am to 6 pm. The length of the shadow changes as the angle of the sun changes in the sky. The length of each of the three objects (tree, person, house) is fixed and cannot be changed.

The user can observe an object's shadow change in size and intensity over the course of a 12 hour period as the sun changes position in the sky. The object's shadow starts at the longest and darkest at the beginning of the day. The shadow gradually decreases in size and shade as noon approaches, where the shadow of the object is at its shortest and lightest in shade. From noon, the shadow gradually increases back to the longest and darkest at the end of the day.

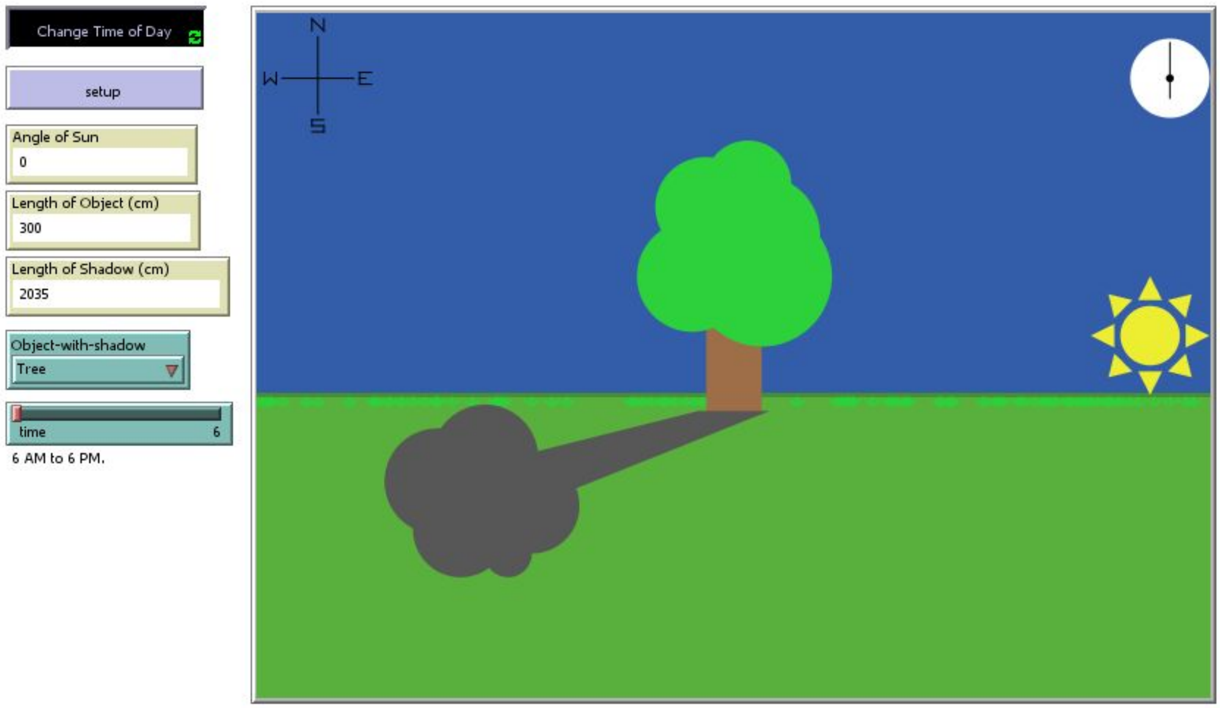


Figure 2: The Shadows and Time simulation interface.

The three investigations were designed to help students move through a progression of three different ways of thinking. First, my goal was to engage students to think of the relationships between the position of the sun and the position of the object's shadow in a non-numeric way. Second, my goal was then to engage students to reason about the relationships in a numeric way; in other words, to use the simulation to determine how one quantity affects the other. Finally, I wanted to explore if students were able to apply the covariation and correspondence knowledge of shadows and time to find measurements of objects or that object's shadow, given the time of day by using ratios and proportional relationships. The following sections describe each of these investigations and my goals in more detail (The full versions of the investigations are presented in the Appendix.)

2.2 Investigation 1: Non-numeric relationships of shadows and time

The goal of this investigation is for the students to determine if a person can tell the time of day from an object's shadow and construct non-numeric relationships about the time of day (angle of the sun) and length of object's shadow. To reach this main goal the investigation has three phases: (1) help the students explore the simulation and identify the variables that vary, (2) help the students explore the difference between a dependent and independent variable, and (3) help the students explore and examine the effects the time of day has on an object's shadow.

The purpose of phase one was to identify the variables of the simulation. By doing this, my goal was for the students to gain knowledge of the program's interface by interacting with the simulation (Figure 3). The purpose of phase two was to distinguish between independent and dependent variables (Figure 4). By doing this, the students were asked to analyze the simulation to determine what they can change and how this influences other variables. The purpose of phase three was to explore the different positions (angles) of the sun at different times of day, and the changes in the shadows (Figure 5). Doing so would allow the students to analyze the relationship between the sun's position and the position of the object's shadow, as well as to identify the factors that change over time (the sun's position, the object's shadow length, and direction) and to construct non-numeric relationships among these variables (Figure 6). These relationships will help the students synthesize the information learned by using the simulation to determine the time of day given an object's shadow.

2. Circle the different variables that can be found in the simulation.
   a. Time
   b. Object with shadow
   c. Length of object
   d. Color of object
   e. Area of shadow
   f. Length of shadow
   g. Light source
   h. Distance of object from the sun
   i. Angle of the sun

Figure 3: Question two from Investigation 1.

5. In the example above, the object is the (dependent, independent) variable and the length of the shadow is the (dependent, independent) variable.

Figure 4: Question five from Investigation 1.

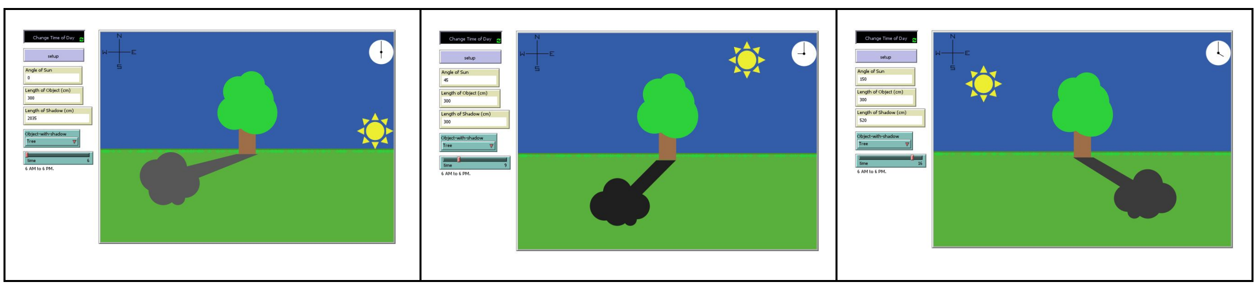


Figure 5: Snapshots of the simulation at three different times of the day.

6. Keep the object the same and manipulate the time slider (independent variable): What happened when you manipulated the time slider? Circle all that apply.

- The sun changed position.
- The sun stayed in the same position.
- The shadow changed position.
- The shadow stayed in the same position.
- The shadow's color stayed the same.
- The shadow's color changed from light to dark.
- The shadow stayed the same size.
- The shadow changed in size.

Figure 6: Question six from Investigation 1.

2.3 Investigation 2: Numeric relationships of shadows and time

In this investigation, the students will explore the numeric relationships of shadows and time. The main goal is for the students to identify the relationships between time and an object's shadow length by reading tables and graphs while looking for patterns in order to make predictions. To reach this goal the investigation has two phases: (1) to complete a table with the length of the tree's shadow from the simulation and the direction in which the shadow falls; (2) to create a graph from the table depicting the relationship between the angle of the sun and the shadow length.

The purpose of phase one was to have the students collect data by observing how the length of shadow changes as the angle of the sun changes and analyze the patterns in the table such as "What is the relationship between the time of day and the length of the shadow?" or "What's the relationship between the shadow length and the sun's angle in the sky?" (Figure 7). The purpose of phase two was to plot ordered pairs from the relationships between time and an object's shadow length and graph the relationship (Figure 8). This will allow the students to interpret relationships in the table and graph, such as "The closer the sun is to the ground, the taller the shadow".

1. Use the tree in the simulation to complete the table below.

Time of Day (angle of sun)	Length of Shadow (cm)	Shadow Direction (N,S,E,W)
Sunrise 15°		
30°		
45°		
60°		
75°		
Noon 90°		
105°		
120°		
135°		
150°		
Sunset 165°		

Figure 7: Question one from Investigation 2.

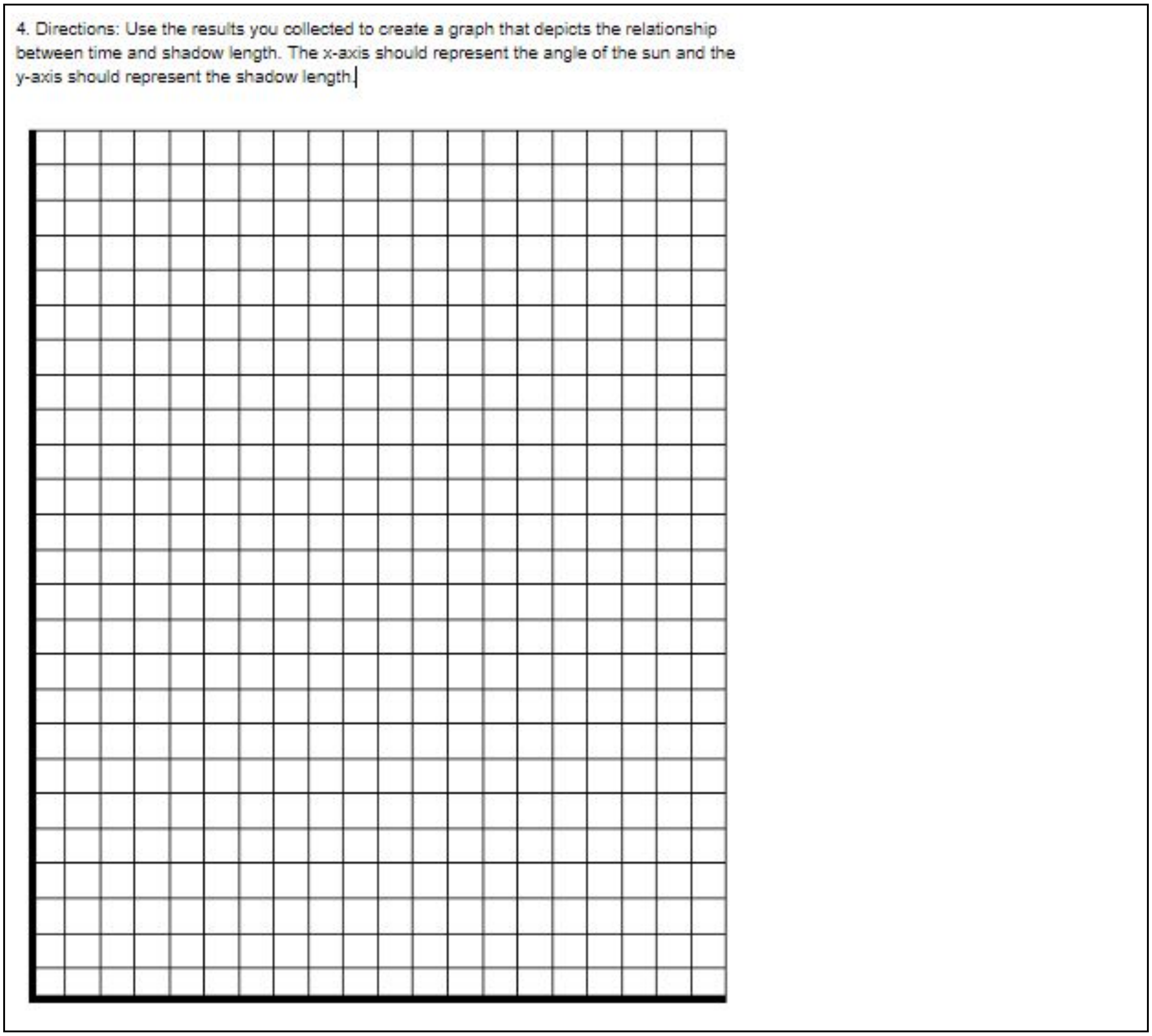


Figure 8: Question four from Investigation 2.

2.4 Investigation 3: Using shadows to find the measurements of other objects

Investigation 3 contains six tasks. In each of the six tasks, the students will use ratios and proportional relationships. To reach this goal the investigation has been broken up into six tasks with three points of focus. The purpose of the three points of focus is for the students to recognize that: (1) at the same point in time, the relationships between two objects and their shadows is the same; (2) the student can use that relationship to solve problems involving given height and shadow length of one object and asking the shadow length of another object given its height; and (3) students can reason about the relationships in tables.

In Task A the students will explore the relationship between the height of a building and its shadow at different times of day, given a table that shows the height and shadow length at both 2 pm and 6 pm (Figures 9a and 9b). By completing the task, the students will analyze the proportional relationships among objects and shadows.

TASK A: The following tables present the height of different buildings and their shadow lengths at different times of the day.

Time: 2 pm	
Height (meters)	**Shadow Length (feet)**
20 (Gym)	10
25 (Bank)	12.5
27 (Restaurant)	13.5
34 (Hospital)	17
40 (Apartment Building)	20

Time: 6 pm	
Height (meters)	**Shadow Length (feet)**
20 (Gym)	40
25 (Bank)	50
27 (Restaurant)	54
34 (Hospital)	68
40 (Apartment Building)	80

Figure 9a: Task A tables from Investigation 3.

1) What is the relationship between the height of the building and the shadow length of the buildings at 2pm?
    a) For each building, the shadow length is double the height of the building.
    b) For each building, the shadow length is half the height of the building.
    c) For each building, the shadow length is 1/3 the height of the building.

2) What is the relationship between the height of the building and the shadow length of the buildings at 6pm?
    a) For each building, the shadow length is double the height of the building.
    b) For each building, the shadow length is half the height of the building.
    c) For each building, the shadow length is 1/3 the height of the building.

3) What did you observe?
    a) The relationship between the height of the buildings and their shadow lengths at a specific time is the same.
    b) The relationship between the height of the buildings and their shadow lengths at a specific time is different.

Figure 9b: Task A questions from Investigation 3.

In Tasks B-F students are asked to model a given situation with a picture in order to find measurements of either an object or its shadow (Figure 10). Modeling each situation can help the students identify the relationship between the objects' sizes and their shadows, as well as the relationship between the two objects that cast their shadows at the same time of day.

TASK F: A tree casts a 2.9-foot shadow at the same time that a 2.4-foot pole casts a 1.3-foot shadow. How tall is the tree? (round your answer to the nearest tenth)

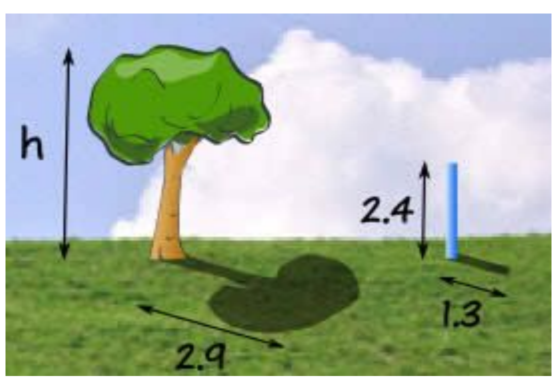


Figure 10: Task F from Investigation 3.

Students are encouraged to reason about both the ***covariation relationship*** (e.g. As the length of object increases by 3, the length of shadow increases by 9) and also the ***correspondence relationship*** (the length of the shadow is 3 times the length of the object) for Tasks B-F. The common goal among all three investigations was to ask questions that would allow the students to use learned knowledge from the simulation, apply that knowledge, and synthesize the knowledge.

2.5 Sample and research context

This study focuses on a teaching experiment with 33 six-graders in a Kearny public school in Hudson County, New Jersey, during the 2017-2018 academic year. The shadows and time module was taught in two classrooms, but for the purpose of this Thesis one class was omitted. The Kearny School district had a total enrollment of 5,045 students at the start of the 2017-2018 academic year. Nearly 60% of the students were Hispanic or African American, and 52% were also classified as economically disadvantaged. The Kearny School district is culturally diverse in population, but they also, categorized as a low performing school district. This performance is measured by the Partnership of Assessment of Readiness for College and Careers (PARCC), which assesses students' (grades k-12) learning of English and Math. The 2016-2017 PARCC academic scores (the academic year before the study) showed that the district was not performing well, with only 40% of their students meeting and exceeding in ELA and 26% meeting and exceeding in Math.

The classroom teacher presented the lessons to maintain the already established classroom atmosphere. Before the teaching experiment, the teacher was provided a detailed module of how each lesson should go, along with a teacher-friendly version of all the student investigations. These versions described the purpose of the investigations and the mathematics they should emphasize with each lesson. The teacher then had the opportunity to ask questions and receive guidance on anything that may have needed clarification. The shadows and time module was designed to help teachers not just understand the material and content of shadows and time, but also for them to truly grasp the concepts in order to guide the students.

The shadows and time module consist of three investigations. Each investigation considered as a lesson covered one class period of 45 minutes over a three day period. Each lesson was video recorded and had an interviewer, who ran the camera and asked the pairs of students questions to check their mathematical thinking. Some of the questions the interviewer would ask are: (1) How can we tell the time of day based on an object's shadow? (2) Which direction will an object's shadow fall in relation to the position of the sun? (3) What are the dependent and independent variable and how do you know? (4) At what times of day will an object's shadow appear the longest? (5) How can we use this table to find the relationship between the height of the building and the shadow length of the building at 2 pm at 6 pm?

These types of questions were chosen first, to uncover the student's mathematical thinking (What have you tried? What happened when you tried it? Why did you try it); second, to examine if the simulation was helpful (What is happening here? What did you notice? When you clicked a button, what did you think would happen?); and finally, to see if the students were able to apply the knowledge learned from the simulation to the investigations.

2.6 Data collection

The primary goal for a design experiment is to improve the initial design by testing and revisiting conjectures as informed by ongoing analysis of both the students' reasoning and the learning environment (Cobb, et al., 2003). The study data comprised video recording, observation notes from classroom interactions (students interviews) during instruction, and students' written work. Also, at the beginning and end of each lesson, there was a debriefing with the teacher to discuss the lesson, the next steps, and things to focus on in the lessons to follow.

2.7 Retrospective analysis

The primary purpose for using video recording to analyze the data is to see students' mathematical learning and reasoning as it happens, and not just to view answers on an assessment. With assessments, there is no way to determine if the students actually knew the answer or simply took a guess. With videos, one can at least see/hear the thought process, even if it does lead to an incorrect answer. Understanding the mathematical process students take when completing a task is powerful. As Steffe and Thompson (2000) argued: "it becomes clear to the researchers who engage in videotape analyses that much of what was learned when working with the students was learned spontaneously and outside their awareness" (p. 292). Student mistakes can be essential to understanding why they made those mistakes in the first place. Steffe and Thompson (2000) stated that when videotapes are carefully analyzed they can offer the opportunity to activate past experiences with the students and bring them to light.

In my analysis of the videos, the goal was to find non-numeric and numeric themes among students generalizations. The generalizations concise of the relationships between objects their shadows and the position of the sun. While looking for generalizations I discovered students attempts to argue why their solution was correct through conversations with an interviewer, classmate or classroom teacher. In order to justify their response, the students had to use prior knowledge as well as knowledge learned in the simulation or in prior investigations. These conversations were very helpful in analyzing the process the students took in determining what they believed to be a correct response. Those types of moments in students thought process could not simply be captured from an assessment.

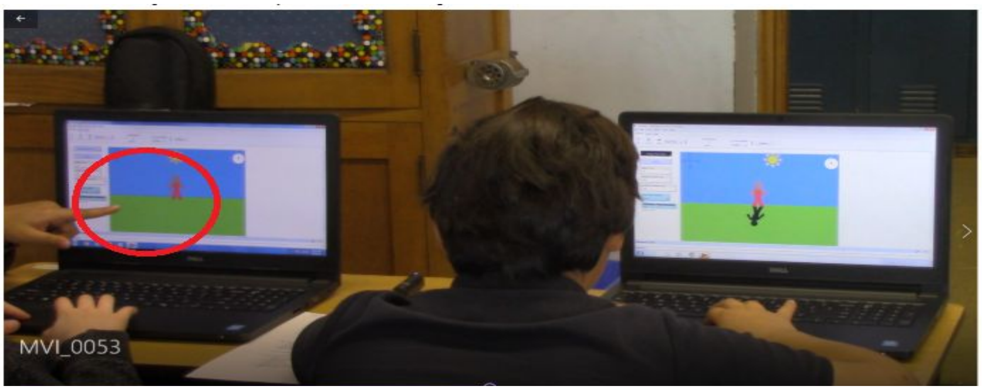
The students written work, investigations, were used to go back to the moments in the video where they made the generalization so I could verify the question at hand as well as the accuracy of the students' generalization. The notes that were taken while viewing the videos were organized in order to analyze the task used to develop students covariational and correspondence relationships and the nature of students reasonings as they engage with those tasks.

3. Findings and Discussion

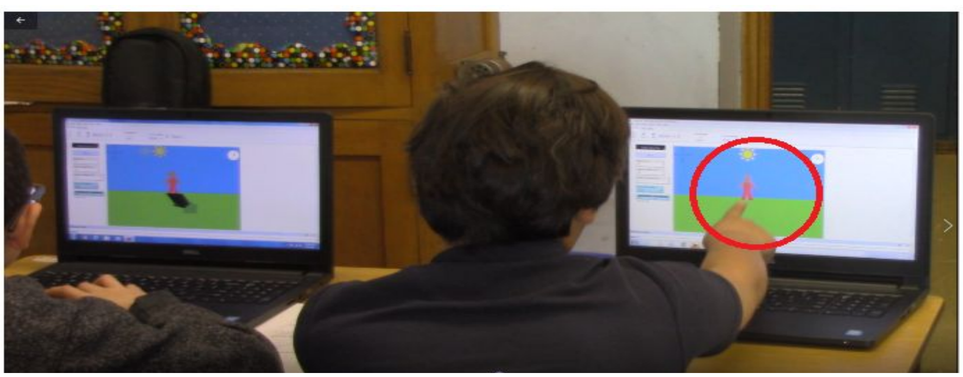
The mathematical thinking that the middle school students experienced in the study of the learning of shadows and time utilizing simulations were characterized by analyzing the video recordings of students working on the investigations. The following subsections describe how students reasoned about (a) the relationship between the position of the sun and shadows and (b) the relationship between the length of the object and its shadow.

3.1 Reasoning about the relationship between the position of the sun and shadows

In Investigation 1, students utilized the simulation to reason about the non-numeric relationship between the angle of the sun and the direction of the object's shadow. The following excerpts are examples of some generalizations the students constructed as they used the simulation to complete various tasks in this investigation.



I: Oh, have you seen he has no shadow, why?
Bill: Because the sun is directly above.
I: If the sun is directly above you don't have a shadow?
Bill: Yeah, you don't have a shadow when the sun is directly above you, because it's just way above. And there is no angle.
I: There's no angle?
Bill: Yeah!
I: Which one is the angle, can you show me on the screen? [Bill uses his finger to point at the



foot of the image of the person to indicate the person is standing on their shadow so no shadow is visible.]
Bill: [Discussing the angle of the sun]Like right here there's an angle, right here these are **all angles that make the shadow longer or shorter**, but here there's no angle going down like

there's nothing like slanted or anything. It's just going straight down so there won't be a shadow. I: So, if we have an angle then we'll have a shadow? Bill: No, no angle, no shadow. I: Do you agree with him? Adam: Yeah, see look if I move slightly towards [cross talk], when it's noon, it's probably straight cause the sun is like beaming at you and won't make a shadow. Probably when it's one (o'clock) there's a shadow slightly towards you. [cross talk] **The sun is shining at you towards an angle which creates a shadow.**

Figure 11: Excerpt one of the interview conversations; Bold font added for emphasis.

Although the students do not see a shadow in the simulation, the sun always casts a shadow. In the case above, the shadow is very small hidden under the object. Due to the limitations of the NetLogo simulation, the very small shadows that exist were not visible on the screen. When the sun is above an object that is 300 units in height forming say a 89° angle (sun altitude) the length of the shadow it will cast is equal to the height of that object divided by Tan(89°) ($L = 300/tan(89)$) . So, if the sun is above the object forming a 89° angle, the length of the shadow (5.24 units) will appear very small relative to the object's height. Excerpt one is an example of how the interviewer prompted the students to discover the relationship between the angle of the sun and the object's shadow. Both students were able to generalize that without an angle (what they identify as an angle of 0 degrees), there will be no shadow. Bill was able to identify that there were two quantities involved: the angle of the sun and the length of the shadow. He also identified that one quantity depends on the other, in other words, the length of the shadow depends on the angle of the sun. Similarly, Adam stated that it is the angle of the sun that creates a shadow so a shadow would depend on that angle.

T: What's the relationship between the shadow length and the sun's angle in the sky
James: **The closer the sun is to the ground the longer the shadow and the farther the sun is from the ground the shorter the shadow.**
…...
I: Do you see any relation between the angles and the length?
Adam: It’s always 15, like 15, 30, 45.
I: So what’s happening with the angle?
Adam: It’s going up by 15.
I: It’s going up by 15.
Bill: And also the shadow,
Adam: It’s going down.
Bill: Decreased. And then after, it increased.

Figure 12: Excerpt two of the interview conversations; bold font added for emphasis.

As the excerpt above illustrates, students identified that the closer the sun is to the ground, the longer the shadow. They also identified the direction of change of the two quantities stating that the as the angle is increasing by 15 degrees, the length is decreasing.

Another important aspect of Investigation 1 was for the students to determine if the time of day could be found from the position of the sun and an object’s shadow. Excerpt three (Figure 13) is what one student discovered.

I: So, what are the things you have noticed when you changed the time slider? What are the things that are changing?
Adam: Uhm, **the sun changes positions. The shadow changes positions.** Uhm, the shadow’s colors.
I: Can we check that?
Adam: The shadows colors changed really dark, cause when far away the light is [cross talk] so it isn’t as dark, as for example, like at one (o’clock), the sun is very close to the house, making the shadow darker, cause more light is entering the area instead of here, because the light can’t actually reach [cross talk] less light reaches the area here [cross talk] then the light was at like I don’t know like 11 or at 1 (o’clock).
I: So, at 11 or 1 (o’clock) the light the darkest?
Adam: The darkest, yeah!

Figure 13: Excerpt three of the interview conversations.

This conversation in Excerpt three between the interviewer and Adam shows that the simulation helped to depict the covariational relationship of that as the sun changes positions, the shadow changes position as well. Additionally, the student recognized the covariational relationship that the closer the sun is to the object, the darker the object's shadow will appear. Which also implies, the farther away the object is from the sun the lighter the object's shadow will appear.

The simulation and careful questioning by the interviewer (e.g. What did you try? What happened then? and Why did it happen?) helped in triggering students' thinking to discover those covariational relationships. By exploring what changes and what stays the same, students were able to generalize how the angle of the sun affects the shade and length of shadows.

The analysis of the next investigation, Investigation 2, focuses on the way the students analyzed the data gathered from the simulation. It shows how students described the patterns they noticed, the numeric relationship between time of day and length of the shadow, and the overall thought processes in creating a graph that depicted the relationship between time of day and length of the shadow. After using the simulation to complete the table (Figure 14a), the following conversations between the interviewer and two different students happened (Figure 14b).

1. Use the tree in the simulation to complete the table below.

Time of Day (angle of sun)	Length of Shadow (cm)	Shadow Direction (N,S,E,W)
Sunrise 15°	**1120**	**West**
30°	**520**	**West**
45°	**300**	**West**
60°	**173**	**West**
75°	**80**	**West**
Noon 90°	**0 (object is on top of shadow)**	**Under object**
105°	**80**	**East**
120°	**173**	**East**
135°	**300**	**East**
150°	**520**	**East**
Sunset 165°	**1120**	**East**

Figure 14a: Question one from Investigation 2 answer key.

I: What do you notice? Do you notice anything on the table?
Adam: You see, umm, when it's on an angle, the angles are the same but direction is different, [cross talk] cause the sun goes in like in a similar direction, on each angle no matter what, see. So it would start here then move towards the left. And then when it moves the shadow up here on the other side of the object. Even though the sun is on a different direction the angle is still the same [cross talk]...
I: And where's the shadow each time?
Adam: **Shadow is on the opposite direction.**

Figure 14b: Excerpt four of the interview conversations.

In the excerpt above (Figure 14b) Adam tries to describe the symmetric relationship he noticed from the table. Similarly, another student, Kevin, argued that although angles might be different "The length is the same from both directions." The conversation implies that students noticed that the angle of the sun increases in intervals of 15° and when two angles of the sun sum up to 180° the length of the object's shadow at those positions will be the same and in opposite directions. These conversations also show how the students were able to correctly determine and describe the path the sun takes from sunrise to sunset. More generally, the sun mirrors its path from sunrise to sunset, meeting in the middle at noon. This allows the shadow of an object to mirror itself as well: From sunrise to noon to sunset, the object's shadow will be the same length, shade, and shape, creating a U-shape pattern. The path the sun takes is symmetric from sunrise to sunset. Say for example, when the sunrise it forms a 15° angle by sunset it will form an angle of 165°and the shadow lengths at those angles will be the same. This will happen for all the *supplementary angles*. The students were 6 graders so some mathematical concepts were not introduced to avoid any confusion. The simulation did those calculations for them.

Figure 15 shows how another student, Heather, reasoned about this pattern she noticed from the table. Although we did not expect her to reason about curves, she was able to discuss how the "numbers are the same on each side" giving the example of the length of shadow being the same at 75° and 105°.

Teacher: What patterns do you see above?
Heather: Umm, it's kind of like, the zero is the middle and umm, the numbers are the same on each side. So, for example, the 80 is on the bottom [cross talk]... So, when the length of the shadow is 80 it's at 75° but at the length of the shadow at 105° it's also 80.

Figure 15: Excerpt five of the interview conversation.

When I asked the students to make a conjecture of how the graph would look like, one student, George, argued that it is not going to be a straight line (Figure 16),. which implies he knows that the relationship was not going to be linear.

I: What do you think, will you get a straight line?
George: No, it may look like a bowl or something.

Figure 16: Excerpt six of the interview conversations.

This investigation showed that by exploring the patterns on the table, students were able to describe the non-linear relationship between the position of the sun and the length of shadow in terms of symmetry and generalize that when we graph this relationship it will not be a straight line. Reasoning in terms of symmetry was also noted after students graphed the relationship. Even though the students were six-graders and only had experience with graphs that showed linear relationships, Excerpt 7 (Figure 17) shows an example of how a student was able to generalize the characteristics of a nonlinear relationship in terms of symmetry.

I: What shape do you think it will be?
Adam: No, I think it'll be like symmetrical on the side cause it's 1120 and 1120 so it's the same thing.

Figure 17: Excerpt seven of the interview conversations.

Not familiar with U-shaped curves, Adam was able to use the data presented in the table to determine that the graph would not be a straight line in his words, “symmetrical”. Next, students graphed this relationship.

After creating the graph, the following conversations happened (Figure 18):

I: But why did we have, why did we have a U, the U-shape?
Sam: Because it was symmetrical it went from up to down and back up again.

Figure 18: Excerpt eight of the interview conversations.

The excerpt above illustrates, how Sam was able to in his own words explain why the graph was not a straight line by describing the behavior of the graph. The students may not have been familiar with graphing curves or their characteristics but they had prior knowledge of linear graphs and their relationship. The graph Sam was able to create did not have the same characteristics of a linear graph, so he knows it was different because his graph, in his words, "was symmetrical it went from up to down and back up again".

In the following excerpt a student, Sandy, explains how the graph could be used to reason about the longest and shortest shadow:

I: So what time of the day is shadow the longest?
Sandy: 6 AM and 6 PM.
I: Where do you see it on the graph?
Sandy: [student points to paper] [Sandy points to the upper left of her graph and then to the upper right of her graph. She is indicating that the object's shadow is at its' longest at sunrise and sunset.] Well here this would be 6 AM and here this would be 6 PM.
I: Yeah, you see that the highest is here, right? And when is the shortest?
Sandy: [Cross talk, both students] Noon.
I: So, can you explain me the graph, like if somebody, can you explain the graph….[Cross talk]....like to someone who doesn't know what's happening, can you explain the graph?
Sandy: **So it starts at 1120 [Cross talk] it decreases and then increases the same way it decreased.**

Figure 19: Excerpt nine of the interview conversations.

Excerpt nine is an example of a student using the graph to answer questions related to the graph. For example, Sandy used the graph to indicate (point out) the times of day an object's shadow is at its longest and shortest. She also described the graph by stating that it decreases and then increases the same way.

3.2 Reasoning about the relationship between the object length and the length of a shadow

In contrast to Investigations 1 and 2, Investigation 3 did not focus on using the simulation. Instead, students were presented a series of tasks about the height of objects and their shadows at a specific time and were asked to reason about the relationships. Tables were used for students to input values and reason about the relationships. Based on the students' responses it was determined that students were able to understand the correspondence relationship between an object's height and shadow length. Figure 20 shows how Jasmine describes a correspondence relationship from two tables in investigation 3 (Figure 9a presented previously):

T: What is the relationship between the height of the building and the shadow length of the building at 6 PM?
Jasmine: Umm, A, [Cross noise] **for each building, the shadow length is double the height of the building.**

Figure 20: Excerpt ten of the interview conversations; bold font added for emphasis.

Excerpt ten is an example of a student describing the correspondence relationship between the height of a building and its' shadow length (Figure 9a) at 6 pm. Jasmine noticed that if she doubled each build's height she would get the shadow length. Similarly, while working on a different task (Figure 21a) John described a correspondence relationship (Figure 21b).

Complete the following table to find your answer.

Length of object (feet)	Length of shadow (feet)
3	9
6	18
9	
90	
	300

Figure 21a: Task B table from Investigation 3.

T: What's the relationship between the length of the shadow and the length of the object?
John: Oh, **the length of the shadow is three times the length of the object.**

Figure 21b: Excerpt eleven of the interview conversations.

In excerpt ten Jasmine was able to describe the relationship by relating the length of the object (height) to the shadow length. In excerpt eleven, John describes a correspondence relationship by relating the shadow length to the length of the object (height).

Students reasoned about the correspondence relationship found in tables by looking across quantities in both directions, from left to right and right to left. Even though through questions we tried to have the students reason covariationally, the correspondence relationship they formed were strong and we noticed that it was hard for them to reason covariationally. Excerpt 12 (Figure 22) is an example of that.

T: Which of the following statements are true?
Kevin: As the length of the object increases by 3, the length of the shadow increases by 3.
T: Is that true? Try it, prove it to me. Show me.
Kevin: So, the line from the object 9 to 27 is increased by 3.
T: So, increased mathematically, when you hear the word increased, right, what are we doing?
Kevin: Multiplying!
T: Are you sure?
James: Adding!
T: Right!

Figure 22: Excerpt twelve conversation between the classroom teacher and student.

Excerpt 12 is an example of how a student reasoned using a correspondence relationship and not the covariation relationship. The students were asked to use a table (Figure 21a presented previously) to make the connection, (as the length of the object {blank} the length of the shadow {blank}). Kevin replies with "as the length of the object increases by 3, the length of the shadow increases by 3” the relationship he is describing is the correspondence relationship across the two quantities. This indicates that the student may not have understood the difference between the covariation relationships and the correspondence relationships. In this case, because the first student gave an incorrect response and was corrected by the teacher there was no way to determine if other students had the correct response or the same reasoning as Kevin. Their difficulty in reasoning covariationally may be due to the fact that the specific investigation did not provide a dynamic simulation to represent those relationships rather they were presented with static tables. Also, there weren’t that many values presented. It’s easier to see the correspondence relationship from the incomplete table. Reasoning about the relationships in a table may have promoted this correspondence type of thinking for the students.

4. Discussion and Concluding Remarks

This study shows that if students engage in exploring the scientific phenomenon of shadows using an interactive simulation, they were able to visualize how shadows change dynamically, identify the factors that lead to these changes and reason about the relationships involved. As the students engaged with the designed tasks and simulation, they were able to extend their knowledge about shadows from being a scientific scenario into a phenomenon which they can reason mathematically. This study explored a) the nature of students’ reasoning about covariational and correspondence relationships as they engage with those tasks and tools and b) the nature of tasks and tools that can be used to develop students’ covariational and correspondence relationships in the context of shadows learning.

In terms of research question a), I found that students used the NetLogo simulation to explore the object's shadows length and the direction of the shadow and reason about the covariational relationship between the time of day (angle of the sun) and an object's shadow. Using the NetLogo simulation students discovered the relationship between the length of the shadow and the shadow's direction, stating that these shadows will always fall in the opposite direction of the sun. The simulation helped the students visualize the characteristics of a shadow namely change in size, shade, as well as the direction extending their knowledge of shadows.

After exploring the relationship of the time of day and the length of an object's shadow, the students created a graph and reasoned about the U-shape as a symmetrical relationship. While creating the graph students discovered the closer the sun is to the ground, the taller the shadow. As well as, the further the sun is from the ground, the shorter the shadow. These discoveries describe the correlational relationship between the object's shadow length and the sun's angle in the sky. Next, students were asked to describe the correspondence and covariational relationship between the height of objects and their shadows. By using the data from the table, students were able to describe the correspondence relationship using variables (letters) or words, for example, if S= length of the shadow and O=length of the object (S=3(O)).

In terms of research question b), this study found that the use of a simulation was helpful in giving the students a visual that was also interactive. Results of this study describe the students' engagement with the simulation and investigations, students were engaged in the learning process. Using a NetLogo simulation, the students were able to reason that the shadow of an object changes in length and shade over the course of a day as well as its position, this is due to the position of the sun. Students described the position based on the direction of the sun. Students also argued that an object's shadow will always fall in the opposite direction of the sun, which can be used to roughly determine the time of day.

Manipulating the time slider in the NetLogo simulation students were able to see how the suns position changed, which affected the shadows; position, color, and size. Identifying this relationship, the characteristics of a shadow depends on the time of day, students were able to discover the non-numeric covariational relationships of shadows and time. The tables in Investigation 3 helped students to reason about the correspondence relationships between the objects' length and its shadow length at a specific time.

The questioning that the teacher and interviewer used helped the students reason about the covariational and correspondence relationships that underlie the concept of shadows. For example, I noticed that questions such as "How would you explain the graph to someone who is not here?" or "What do you notice?" helped to uncover the students' thinking by forcing them to describe their thought process. Questions such as "Where do you see it on the graph?" or "What shape do you think it will be? " helped to focus the students allowing them to apply their own hypothesis to each question. Questions such as the following are important because, the students will have to justify their responses, which implies the responses are of their own; "Is that true? Try it, prove it to me. Show me.". This pushed the student to explain why the plotted points on the graph created the specific shape.

To sum up, based on the results the study was successful in developing students' covariational and correspondence reasoning. With further use of simulations showcasing the natural phenomena, students may be able to gain a better understanding of the mathematical relationships that exist within the phenomena.

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6. APPENDICES

Appendix A: Investigation 1: Non-numeric relationships of shadows and time

Appendix B: Investigation 2: Numeric relationships of shadows and time

Appendix C: Investigation 3: Using shadows to find the measurements of other objects

APPENDIX A

Investigation 1: Non-numeric relationships of shadows and time

Terms we will be using in this investigation

A **variable** is a factor that can be changed or controlled. A variable can be independent or dependent.

An **independent variable** is a factor that is changed by the scientist.

- What is tested
- What is manipulated

A **dependent variable** is a factor that might be affected by the change in the independent variable.

- What is observed
- What is measured
- The data collected

SET UP: Please follow the following instructions to set up the simulation:

1. Open the Shadows and Time Simulation by double-clicking on the NetLogo icon.
2. Choose an object from the drop-down panel labeled “Object with shadow”. (tree, house or person)
3. Click the “Setup” button.
4. Click the “Change Time of Day” button. Simulation is ready to explore! (**Note: If you decide to change from one object to another, after selecting the new object, you MUST click “Setup” again**)

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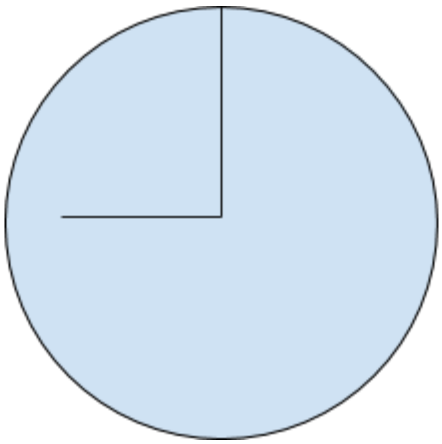
1. Take 5 minutes to explore how the simulation works.

2. Circle the different variables that can be found in the simulation.
    a. Time
    b. Object with shadow
    c. Length of object
    d. Color of object
    e. Area of shadow
    f. Length of shadow
    g. Light source
    h. The distance of the object from the sun
    i. The angle of the sun

3. Move the time slider until the clock in the simulation shows the following:



Is this morning or afternoon? What time is this?

____ o'clock in the (morning / afternoon)

4. Keep the time the same and switch to different objects. What happens?
- The length of the shadow is changing.
- The length of the shadow is not changing.

5. In the example above, the object is the (dependent, independent) variable and the length of the shadow is the (dependent, independent) variable.

6. Keep the object the same and manipulate the time slider (independent variable):
What happened when you manipulated the time slider? Circle all that apply.
- The sun changed position.
- The sun stayed in the same position.
- The shadow changed position.
- The shadow stayed in the same position.
- The shadow's color stayed the same.
- The shadow's color changed from light to dark.
- The shadow stayed the same size.
- The shadow changed in size.

7. In the example above, the time is the (dependent, independent) variable and the position of the sun, the position of the shadow and the shadow's size and color are the (dependent, independent) variables.

8. Manipulate the variables to answer the following questions:

a) The objects cast the shortest shadow at ____ pm.

b) The objects cast the longest shadow at ____am and ____ pm.

c) When the sun (or any light source) is on the east of the object, the direction of the shadow is:
   i) North
   ii) South
   iii) East
   iv) West

d) If you are the object and the Sun is behind you, where is your shadow falling?
   i) behind me
   ii) in front of me
   iii) to my right side
   iv) to my left side
   v) no shadow

e) When the sun (or any light source) is on the east of the object, the shadow is (<u>smaller / larger)</u> than the object.

f) If you are the object, and the Sun is directly above your head, where is your shadow?
   i) behind me
   ii) in front of me
   iii) to my right side
   iv) to my left side
   v) below me

g) When the sun (or any light source) is west of the object, the direction of the shadow is:

i) North
ii) South
iii) East
iv) West

h) When the sun (or any light source) is west of the object, the shadow is (smaller / larger) than the object.

i) When the sun (or any light source) is directly above the object, what happens to the shadow and its length? (Tick all that apply)

i) The shadow is on the east of the object.
ii) The shadow is on the west of the object.
iii) The shadow is below the object.
iv) The shadow is directly in front of the object.
v) The shadow is smaller than the object.
vi) The shadow has the same length as the object.
vii) The shadow is longer than the object.

j) Changing the following will make an object's shadow bigger (Tick all that apply):

i) The time
ii) The object
iii) The position of the sun
iv) The object's color

k) At which angles of the sun is the shadow length equal to the length of the object?

i) 10 degrees and 150 degrees
ii) 10 degrees and 180 degrees
iii) 45 degrees and 135 degrees
iv) 60 degrees and 75 degrees
v) 105 degrees and 20 degrees

l) At which times of the day is the shadow length and object length the same?

i) 6am and 9am

ii) 6am and 1pm

iii) 9am and 3pm

iv) 9am and 4pm

v) noon and 6pm

m) When is the shadow the shortest?

i) 8am

ii) 10am

iii) noon

iv) 2pm

v) 4pm

APPENDIX B

Investigation 2: Numeric relationships of shadows and time

1. Use the tree in the simulation to complete the table below.

Time of Day (angle of sun)	Length of Shadow (cm)	Shadow Direction (N,S,E,W)
Sunrise 15°		
30°		
45°		
60°		
75°		
Noon 90°		
105°		
120°		
135°		
150°		
Sunset 165°		

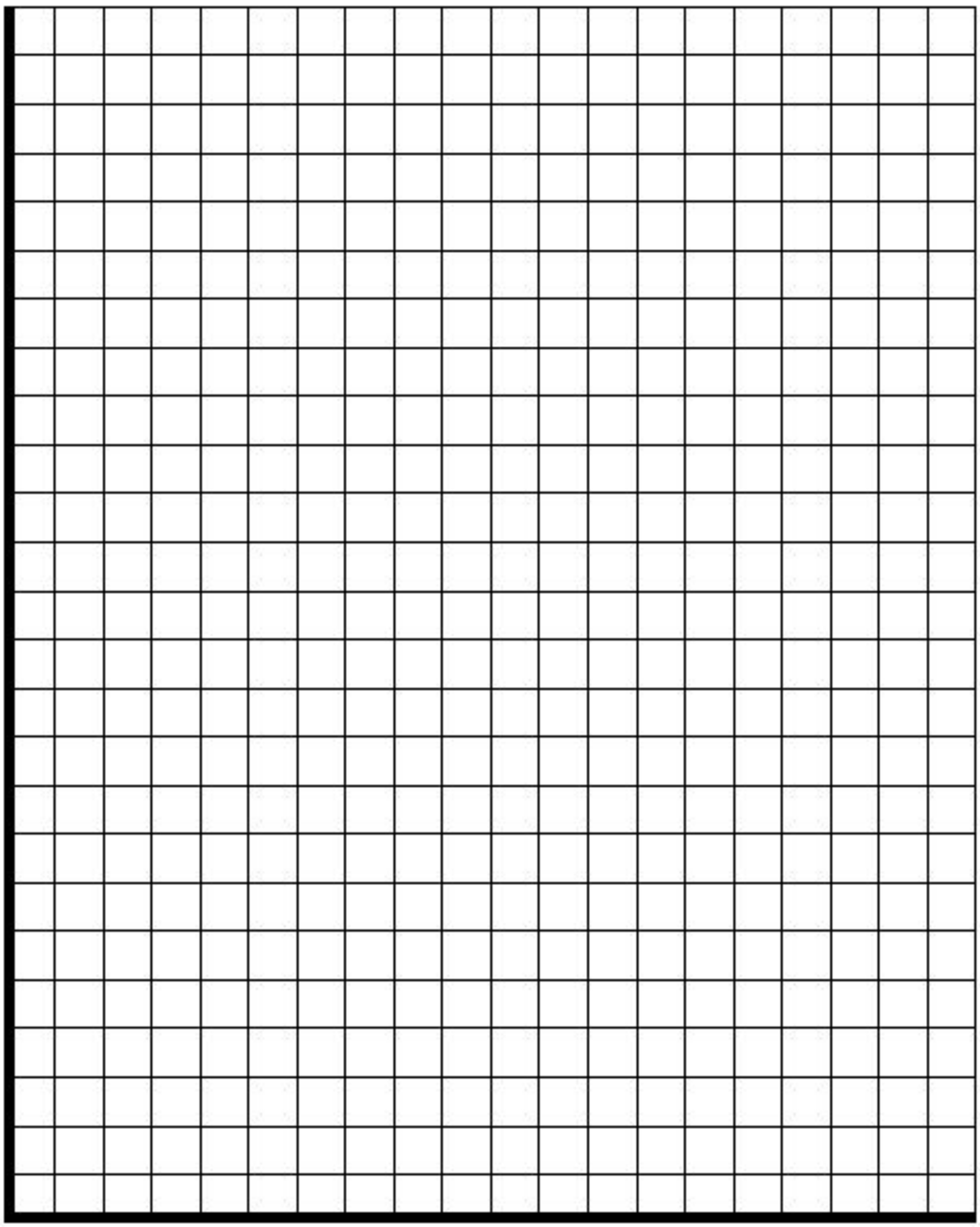
2. What patterns do you see in the table above? Circle all that apply.

- From sunrise until noon, the angle of the sun increases while the length of shadow decreases; from noon to sunset, the angle of the sun increases while the length of the shadow increases.
- From sunrise until noon, the angle of the sun increases while the length of shadow increases; from noon to sunset, the angle of the sun increases while the length of the shadow decreases.
- The direction of the shadow is always in the opposite direction of the sun.
- The direction of the shadow is always in the same direction as the sun.

3. Which of the following describes the relationship between the time of day and the length of the shadow.

- The length of the shadow changes 15 cm every hour.
- The length of the shadow changes every 15°.
- The direction of the shadow changes every 15°.
- The direction of the shadow changes every 15 cm.

4. Directions: Use the results you collected to create a graph that depicts the relationship between time and shadow length. The x-axis should represent the angle of the sun and the y-axis should represent the shadow length.



Use the graph to respond to the following questions:

a) Which of the following describes what the graph shows.

- A straight line (going up from left to right)
- A straight line (going down from left to right)
- A U-shape curve (open upwards)
- A U-shape curve (open downwards)

b) What time of day was the shadow the longest?

- 3 am and 3 pm
- 6 am and 6 pm
- 3 am and 6 pm
- 6 am and 3 pm

c) What time of day was the shadow the shortest?

- 6 am
- 8 am
- 10 am
- Noon
- 2 pm
- 4 pm
- 6 pm

d) What's the relationship between the shadow length and the sun's angle in the sky? Circle all that apply.

- The closer the sun is to the ground, the taller the shadow.
- The further the sun is to the ground, the taller the shadow.
- The closer the sun is to the ground, the shorter the shadow.
- The further the sun is from the ground, the shorter the shadow.

e) What's the relationship between the direction in which a shadow falls and the location of the light source?

- The shadow falls on the same side as the sun.
- The shadow falls in the opposite direction of the sun.

APPENDIX C

Investigation 3: Using shadows to find the measurements of other objects

TASK A: The following tables present the height of different buildings and their shadow lengths at different times of the day.

Time: 2 pm	
Height (meters)	**Shadow Length (feet)**
20 (Gym)	10
25 (Bank)	12.5
27 (Restaurant)	13.5
34 (Hospital)	17
40 (Apartment Building)	20

Time: 6 pm	
Height (meters)	**Shadow Length (feet)**
20 (Gym)	40
25 (Bank)	50
27 (Restaurant)	54
34 (Hospital)	68
40 (Apartment Building)	80

1) What is the relationship between the height of the building and the shadow length of the buildings at 2 pm?
   a) For each building, the shadow length is double the height of the building.
   b) For each building, the shadow length is half the height of the building.
   c) For each building, the shadow length is 1/3 the height of the building.

2) What is the relationship between the height of the building and the shadow length of the buildings at 6 pm?
   a) For each building, the shadow length is double the height of the building.
   b) For each building, the shadow length is half the height of the building.
   c) For each building, the shadow length is 1/3 the height of the building.

3) What did you observe?
   a) The relationship between the height of the buildings and their shadow lengths at a specific time is the same.
   b) The relationship between the height of the buildings and their shadow lengths at a specific time is different.

TASK B: If a 3ft tall dog casts a shadow of 9ft, how long is the shadow of a 9ft elephant?

1) Draw a picture to model this problem. How can you use the picture to find the answer?

2) Complete the following table to find your answer.

Length of object (feet)	Length of shadow (feet)
3	9
6	18
9	
90	
	300

3) Which of the following statements is true?
- As the length of the object increases by 3, the length of shadow increases by 3.
- As the length of the object increases by 3, the length of shadow increases by 9.
- As the length of the object increases by 3, the length of shadow decreases by 3.
- As the length of the object increases by 3, the length of shadow decreases by 9.

4) What’s the relationship between the length of the shadow and the length of the object?
   - The length of the shadow is 2 times the length of the object.
   - The length of the shadow is 3 times the length of the object.
   - The length of the object is 2 times the length of the shadow.
   - The length of the object is 3 times the length of the shadow.

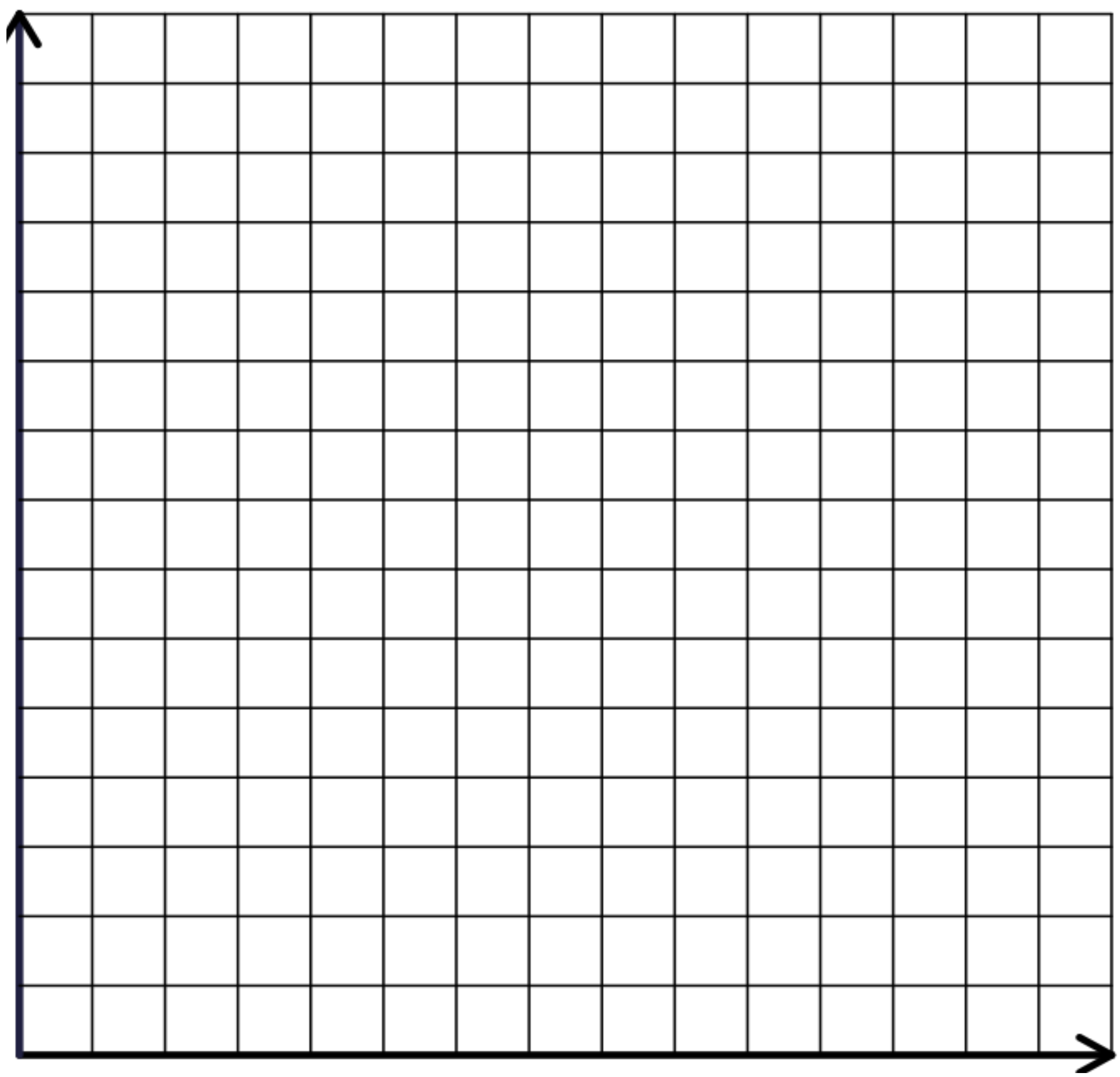
5) Which of the following describes the relationship using variables (letters) or words. If S= length of the shadow and O= length of the object.
   - S= 2 (O)
   - S= 3 (O)
   - O= 2 (S)
   - O= 3 (S)

6) Find the length of the shadow of a building that is 160 feet.
   - 160 ft
   - 320 ft
   - 480 ft
   - 640 ft

7) Make a graph that describes the relationship between the length of the object and the length of the shadow. Label the axes.

   The x-axis will represent the independent variable, which is:
   - Length of object
   - Length of shadow

   The y-axis will represent the dependent variable, which is:
   - Length of object
   - Length of shadow



8) Which of the following describes the shape of the graph?
   - A straight line (going up from left to right)
   - A straight line (going down from left to right)
   - A U-shape curve (open upwards)
   - A U-shape curve (open downwards)

TASK C: If the Eiffel Tower casts a shadow of 328 ft and a nearby light pole that stands 6 ft tall casts a shadow of 2 ft. How tall is the Eiffel Tower?

1) Draw a picture to model this problem. How can you use the picture to find the answer?

2) Complete the following table to find your answer.

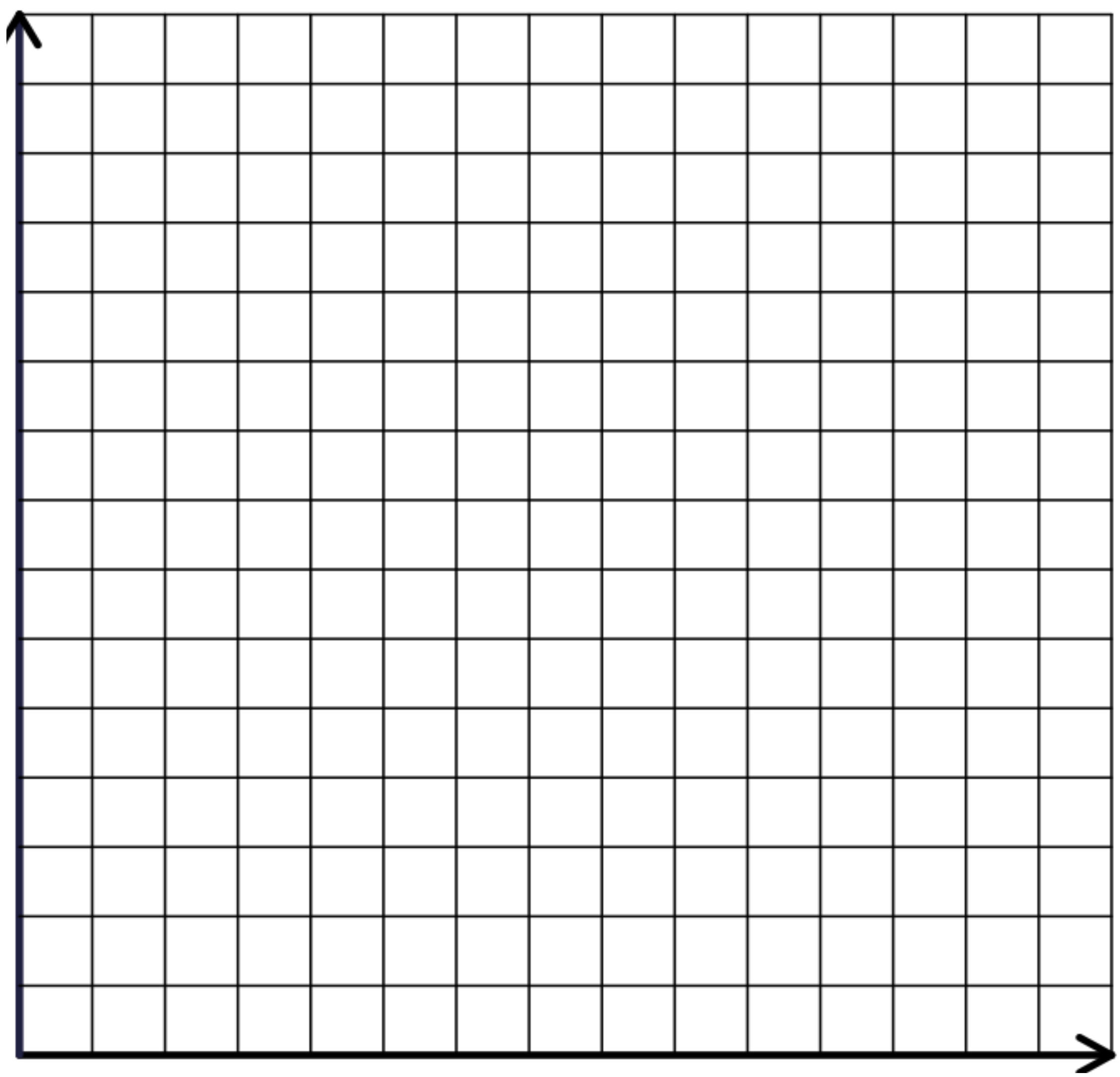
Length of object (feet)	Length of shadow (feet)
6	2
9	3
12	
90	
	300

3) Which of the following statements is true?
   - As the length of the object is increasing by 3, the length of the shadow is increasing by 1.
   - As the length of the object is increasing by 3, the length of the shadow is increasing by 3.
   - As the length of the object is increasing by 3, the length of the shadow is decreasing by 1.
   - As the length of the object is increasing by 3, the length of the shadow is decreasing by 3.
4) What's the relationship between the length of the shadow and the length of the object?
   - The length of the shadow is 1/3 times the length of the object.
   - The length of the shadow is 3 times the length of the object.
   - The length of the object is 1/3 times the length of the shadow.
   - The length of the object is 3 times the length of the shadow.
5) Which of the following describes the relationship using variables (letters) or words. If S= length of the shadow and O= length of the object.
   - S= 3 (O)
   - S= 1/3 (O)
   - O= 2 (S)
   - O= 1/3 (S)
6) Find the length of the shadow of a statue that is 30 feet.
   - 5 ft
   - 10 ft
   - 15 ft
   - 20ft

7) Make a graph that describes the relationship between the length of the object and the length of the shadow. Label the axes.

    The x-axis will represent the independent variable, which is:
    - Length of object
    - Length of shadow

    The y-axis will represent the dependent variable, which is:
    - Length of object
    - Length of shadow



TASK D: A 8 ft tall telephone booth in the Zoo, outside of the giraffe enclosure casts a 4 ft shadow. If the giraffe cast a 7 ft long shadow how tall is the giraffe?

1) Draw a picture to model this problem. How can you use the picture to find the answer?

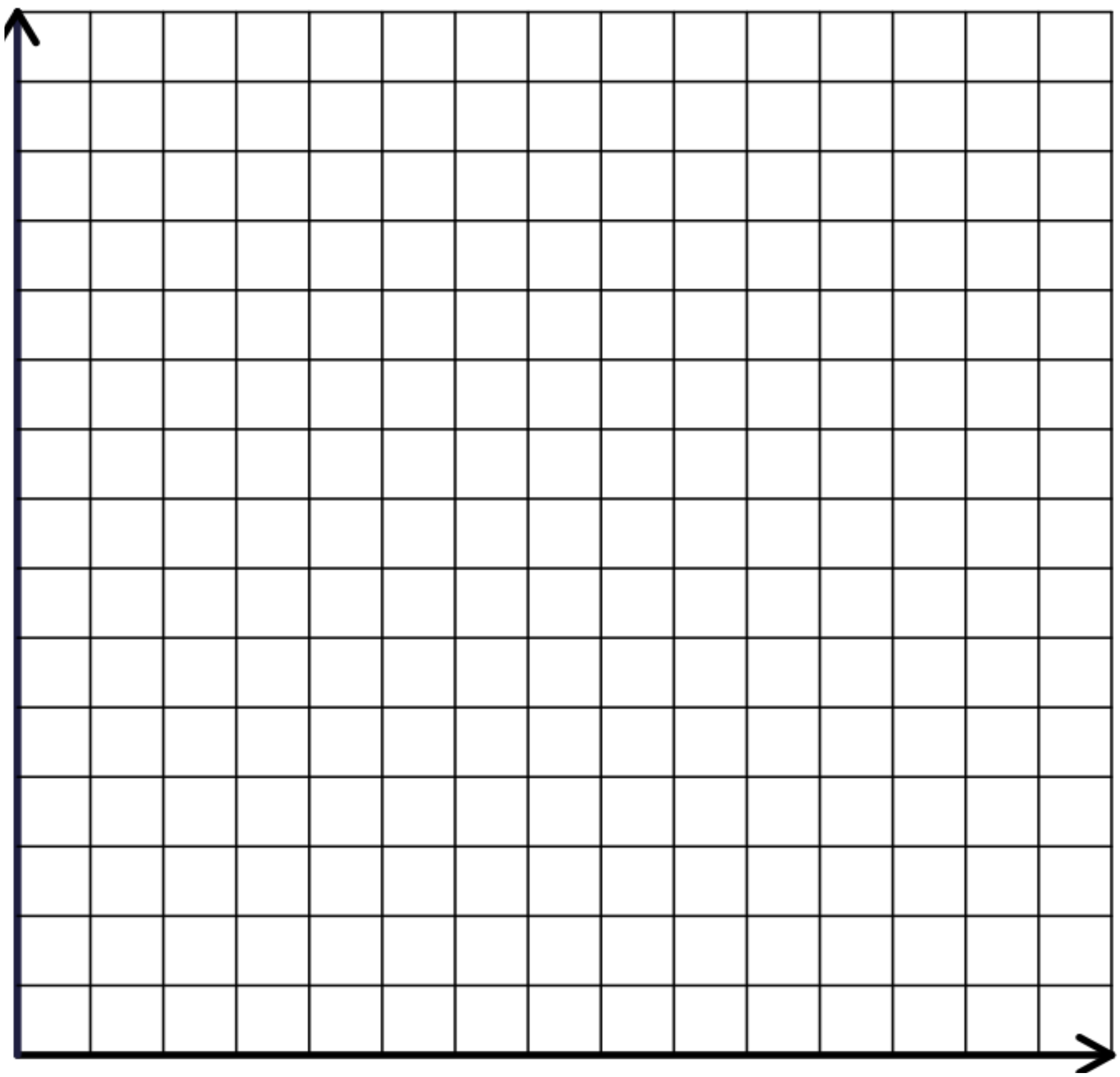
2) Complete the following table to find your answer.

Length of object (feet)	Length of shadow (feet)
8	4
16	8
	1
	7
80	
	300

3) What's the relationship between the length of the shadow and the length of the object?
   - The length of the shadow is 1/3 times the length of the object.
   - The length of the shadow is 1/2 times the length of the object.
   - The length of the object is 1/3 times the length of the shadow.
   - The length of the object is 1/2 times the length of the shadow.

4) Which of the following describes the relationship using variables (letters) or words. If S= length of the shadow and O= length of the object.
   - S= ⅓ (O)
   - O= ⅓ (S)
   - O= ½ (S)
   - S= ½ (O)

5) Make a graph that describes the relationship between the length of the object and the length of the shadow.



6) Use the graph to find the length of the shadow of a tree that is 75 feet.
    - 37.5 ft
    - 25 ft
    - 15ft
    - 7.5 ft

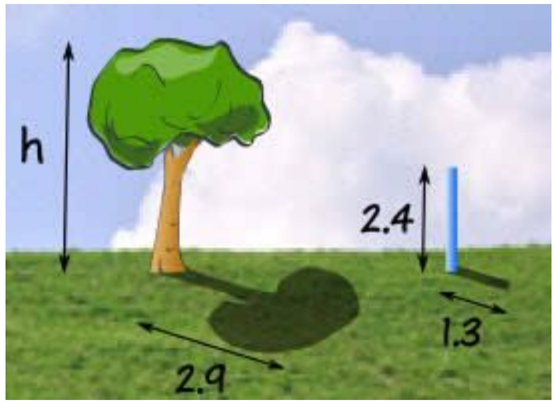
TASK E: Your classroom globe that is 3 ft tall casts a shadow that is 7 ft long. Find the length of your shadow. (if you do not know your height use your teacher's height)?

1) Draw a picture to model this problem. How can you use the picture to find the answer?

2) Complete the following table to find the shadow lengths of different people.

Length of object (feet)	Length of shadow (feet)
3	7
4.5	
6	
6.6 (a basketball player)	
7.8 (the world's tallest woman)	
_____ (your height)	

TASK F: A tree casts a 2.9-foot shadow at the same time that a 2.4-foot pole casts a 1.3-foot shadow. How tall is the tree? (round your answer to the nearest tenth)



a) Complete the following table for different objects and the length of their shadows.

Length of object (feet)	Length of shadow (feet)
2.4	1.3
	2.9
	5
14.4	

b) Which of the following would describe the relationship between the different objects and the length of their shadows.

- A straight line (going down from left to right)
- A straight line (going up from left to right)
- A U-shape curve (open downwards)
- A U-shape curve (open upwards)